

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \rightarrow a} f(x) = L \quad (6)$$

which is read "the limit of $f(x)$ as x approaches a is L " or " $f(x)$ approaches L as x approaches a ." The expression in (6) can also be written as

$$f(x) \rightarrow L \text{ as } x \rightarrow a \quad (7)$$

The One-sided Limits

$\lim_{x \rightarrow a^-} f(x)$ is called **left limit**

$\lim_{x \rightarrow a^+} f(x)$ is called **right limit**

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$$

Two-sided Limit One-sided Limits

If $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x) \Rightarrow \lim_{x \rightarrow a} f(x) = \text{doesn't exist (DNE)}$

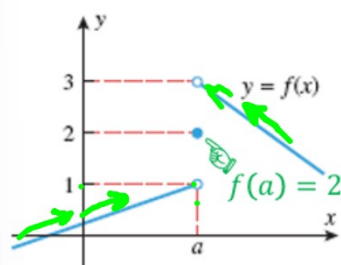
Notation:

- $x \rightarrow a^-$ (x approaches a from the **left**)
- $x \rightarrow a^+$ (x approaches a from the **right**)

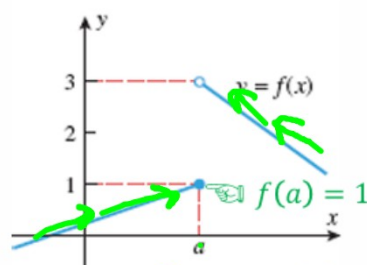
Beware:

$-a \neq a^-$
 $+a \neq a^+$ Only appear in limits

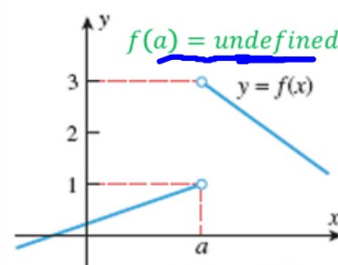
Example 5 For the functions find the one-sided and two-sided limits at $x = a$ if they exist.



$$\begin{aligned} \lim_{x \rightarrow a^+} f(x) &= 1 \\ \lim_{x \rightarrow a^-} f(x) &= 3 \\ \lim_{x \rightarrow a} f(x) &= \text{DNE} \end{aligned}$$

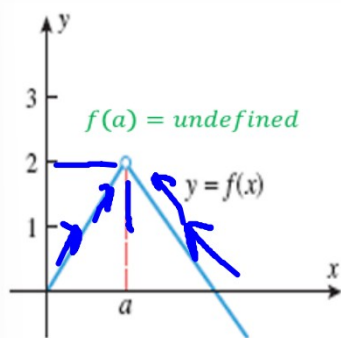


$$\begin{aligned} \lim_{x \rightarrow a^+} f(x) &= 3 \\ \lim_{x \rightarrow a^-} f(x) &= 1 \\ \lim_{x \rightarrow a} f(x) &= \text{DNE} \end{aligned}$$



$$\lim_{x \rightarrow a} f(x) = \text{DNE}$$

Example 6 For the functions find the one-sided and two-sided limits at $x = a$ if they exist.

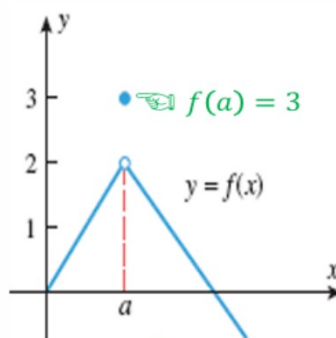


$$\lim_{x \rightarrow a^-} f(x) = 2$$

$$\lim_{x \rightarrow a^+} f(x) = 2$$

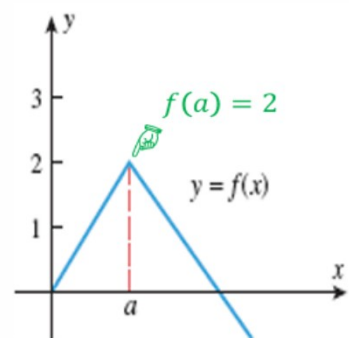
$$\lim_{x \rightarrow a} f(x) = 2$$

$$f(a) = \text{undefined}$$



$$\lim_{x \rightarrow a} f(x) = 2$$

$$f(a) = 3$$



$$\lim_{x \rightarrow a} f(x) = 2$$

$$f(a) = 2$$

Example: Find the limits from the graph

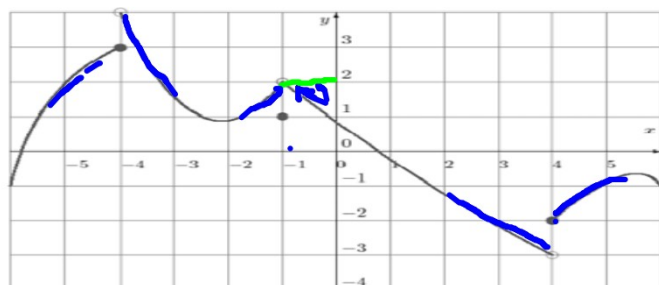
(a) $\lim_{x \rightarrow -1^+} f(x) = 2$

(b) $\lim_{x \rightarrow -1^-} f(x) = 2$

(c) $\lim_{x \rightarrow -1} f(x) = 2$

(d) $\lim_{x \rightarrow -4} f(x) =$

(e) $\lim_{x \rightarrow 4} f(x) =$



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$\lim_{x \rightarrow -4} f(x) = 3$

$x \rightarrow -4^-$

$\lim_{x \rightarrow -4^+} f(x) = 4$

$x \rightarrow -4^+$

$\lim_{x \rightarrow -4} f(x) = DNE$

$x \rightarrow -4$

(e) $\lim_{x \rightarrow -2} f(x) = -2$

$x \rightarrow -2^+$

$\lim_{x \rightarrow -2^-} f(x) = -3$

$x \rightarrow -2^-$

$\lim_{x \rightarrow -2} f(x) = DNE$

$x \rightarrow -2$

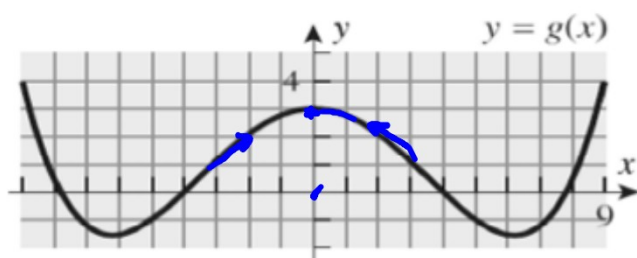
1. For the function g graphed in the accompanying figure, find

(a) $\lim_{x \rightarrow 0^-} g(x) = 3$

(b) $\lim_{x \rightarrow 0^+} g(x) = 3$

(c) $\lim_{x \rightarrow 0} g(x) = 3$

(d) $g(0) = 3$



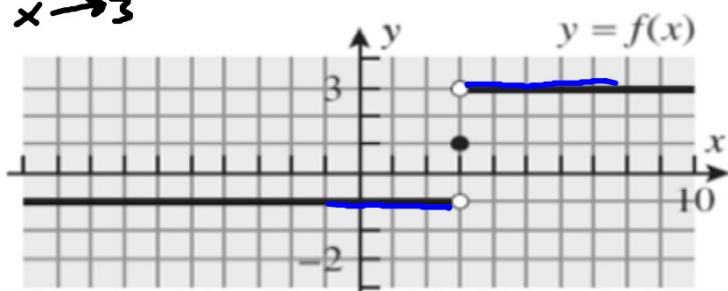
3. For the function f graphed in the accompanying figure, find

(a) $\lim_{x \rightarrow 3^-} f(x) = -1$

(b) $\lim_{x \rightarrow 3^+} f(x) = 3$

(c) $\lim_{x \rightarrow 3} f(x) = \text{DNE}$

(d) $f(3) = 1$



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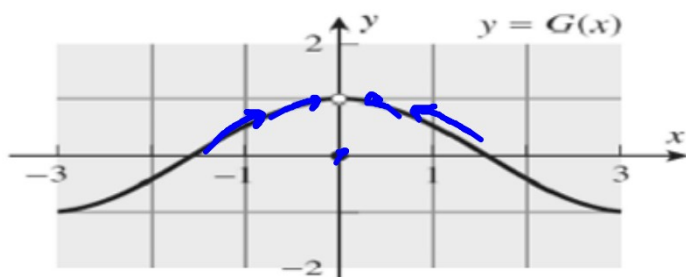
6. For the function G graphed in the accompanying figure, find

(a) $\lim_{x \rightarrow 0^-} G(x) = 1$

(b) $\lim_{x \rightarrow 0^+} G(x) = 1$

(c) $\lim_{x \rightarrow 0} G(x) = 1$

(d) $G(0) = 0$



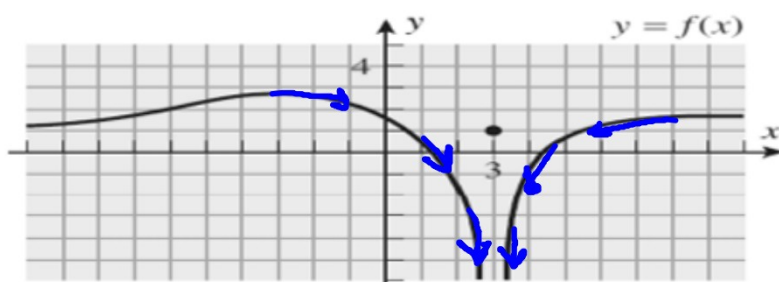
7. For the function f graphed in the accompanying figure, find

(a) $\lim_{x \rightarrow 3^-} f(x) = -\infty$

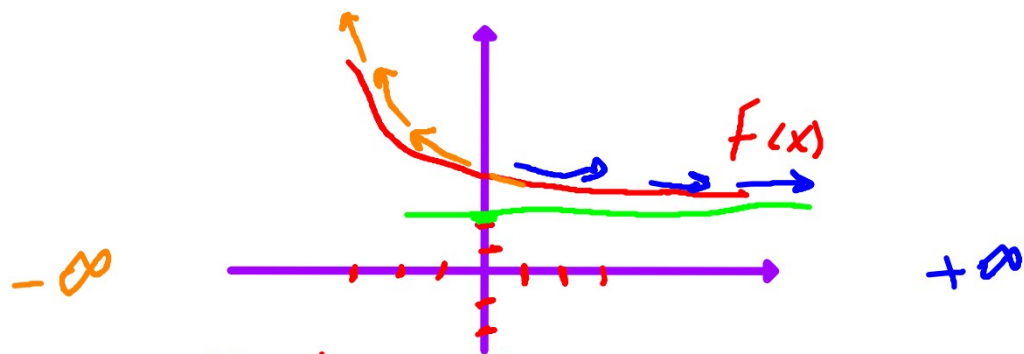
(b) $\lim_{x \rightarrow 3^+} f(x) = -\infty$

(c) $\lim_{x \rightarrow 3} f(x) = -\infty$

(d) $f(3) = 1$



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Find $\lim_{x \rightarrow +\infty} f(x)$

$\lim_{x \rightarrow -\infty} f(x)$

OK!

$\lim_{x \rightarrow +\infty} f(x) = 2$

$\lim_{x \rightarrow -\infty} f(x) = +\infty$

► Example (classic) Find $\lim_{x \rightarrow 0} \frac{1}{x} = \text{DNE}$

Q55:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty \quad \Rightarrow \quad \lim_{x \rightarrow 0} \frac{1}{x} = \text{DNE}$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

Various cases:

$$\lim_{\substack{x \rightarrow a^- \\ x \rightarrow a^+ \\ x \rightarrow a}} f(x) = \begin{cases} -\infty \\ \infty \end{cases}$$

- decrease without bound
- increase without bound



$$\lim_{x \rightarrow a^+} \frac{1}{x-a} = +\infty \quad \text{and} \quad \lim_{x \rightarrow a^-} \frac{1}{x-a} = -\infty$$

ex: $\lim_{x \rightarrow 5^+} \frac{1}{x-5} = +\infty$, $\lim_{x \rightarrow 5^-} \frac{1}{x-5} = -\infty$
 $\Rightarrow \lim_{x \rightarrow 5} \frac{1}{x-5} = \text{DNE}$

(b)

$$\lim_{x \rightarrow a} \frac{1}{(x-a)^2} = \lim_{x \rightarrow a^+} \frac{1}{(x-a)^2} = \lim_{x \rightarrow a^-} \frac{1}{(x-a)^2} = +\infty$$

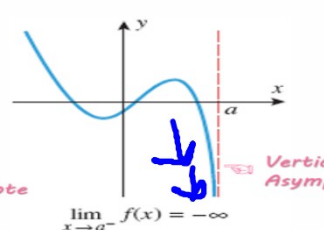
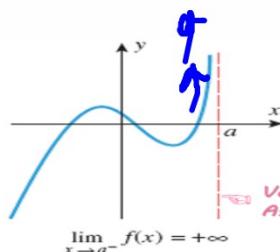
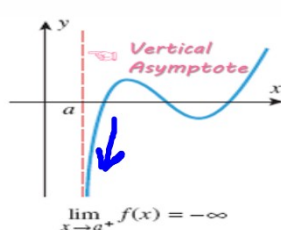
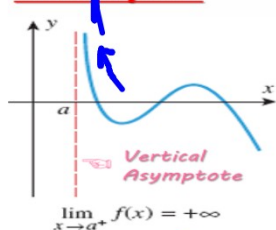
ex: $\lim_{x \rightarrow 3^+} \frac{1}{(x-3)^2} = +\infty$, $\lim_{x \rightarrow 3^-} \frac{1}{(x-3)^2} = +\infty$

Vertical Asymptotes:

A vertical line $x = a$ is called a **vertical asymptote** of the graph of a function f if at least one of the following holds

$\lim_{x \rightarrow a^+} f(x) = +\infty$, $\lim_{x \rightarrow a^+} f(x) = -\infty$, $\lim_{x \rightarrow a^-} f(x) = +\infty$, $\lim_{x \rightarrow a^-} f(x) = -\infty$

Example:



$\lim_{x \rightarrow a} f(x) = \pm \infty \Leftrightarrow x = a$ is vertical asymptote

Horizontal Asymptotes:

A horizontal line $y = L$ is called a **horizontal asymptote** of the graph of a function f if either

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

$\lim_{x \rightarrow \pm \infty} f(x) = L \Leftrightarrow y = L$ is horizontal asymptote

ex: if $\lim_{x \rightarrow 3^+} f(x) = +\infty$ then

1) $x=3$ is h.g

2) $y=3$ is h.g

3) $x=3$ is v.g

4) $y=3$ is w.g

ex: if $\lim_{x \rightarrow +\infty} f(x) = 7$

then $x=7$ is v.g

o True o False

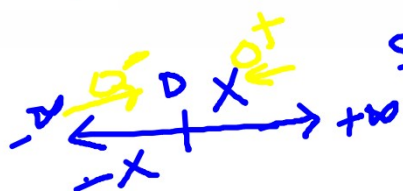
ex: if $\lim_{x \rightarrow +\infty} f(x) = 5$ then

1) $x=5$ is h.g

2) $y=5$ is v.g

3) $y=5$ is h.g ✓

► Example (classic) Find $\lim_{x \rightarrow 0} \frac{|x|}{x}$



def)

$$|x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} \frac{|x|}{x} = \begin{cases} \lim_{x \rightarrow 0^+} \frac{x}{x} = 1 \\ \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1 \end{cases}$$

$$\rightarrow \lim_{x \rightarrow 0} \frac{|x|}{x} = \text{DNE}$$

(4) 5.5

$$\text{ex } \lim_{x \rightarrow 0^+} \frac{|x|}{x}$$

A) +1

B) -1

C) DNE

D) 0

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x}$$

A) +1

B) -1

C) DNE

D) 0

$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

A) +1

B) -1

C) DNE

D) 0

► Example 2 Use numerical evidence to make a conjecture about the value of

$$\lim_{x \rightarrow 1} (3x + 2)$$

1+

x	0.5	0.99	0.999	1	1.001	1.01	1.2
f(x)	—	—	4.99	5	5.001	—	—

$$3(1) + 2 = 5$$

